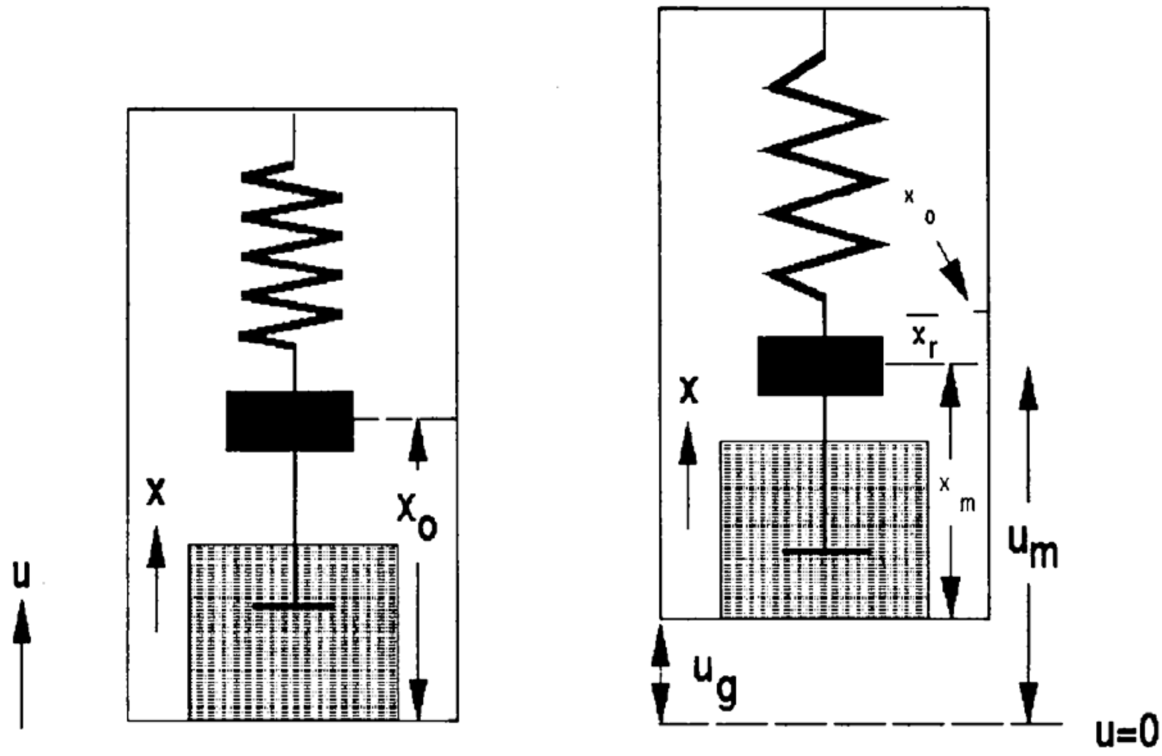
RESTITUTION OF GROUND MOTIONS

Suggested literature:

Scherbaum, F. 1996. Of Zeros and Poles. Fundamentals of Digital Seismology. In 'Modern Approaches in Geophysics', Kluwer Academic Publishers, 256 pages.

THE SEISMOMETER



s0401

Three forces describe the motion of a seismometer:

Inertial force (⇒ acceleration of the ground acting on mass 'm')

$$f_i = -m\ddot{u}_m(t)$$

Frictional force (dashpot ⇒ the velocity of the mass)

$$f_f = -D\dot{x}_m(t)$$

Restoring force (the spring ⇒ displacement of mass)

$$f_{sp} = -kx_r(t)$$

⇓

$$f_i + f_{sp} + f_f = 0$$

$$u_m(t) = u_g(t) + x_m(t) \text{ and } \dot{x}_m(t) = \dot{x}_r(t) \text{ and } \ddot{x}_m(t) = \ddot{x}_r(t)$$

$$m\ddot{x}_r(t) + D\dot{x}_r(t) + kx_r(t) = -m\ddot{u}_g(t)$$

Dividing both sides by the mass 'm' leads to

$$\ddot{x}_r(t) + \frac{D}{m}\dot{x}_r(t) + \frac{k}{m}x_r(t) = -\ddot{u}_g(t)$$

Substituting $\frac{D}{m} = 2h\omega_0$, and $\frac{k}{m} = \omega_0{}^2$, we get

$$\ddot{x}_r(t) + 2h\omega_0\dot{x}_r(t) + \omega_0^2 x_r(t) = -\ddot{u}_g(t)$$

Note, that '*h*' is referred to as the 'damping constant' of the instrument,
(the 'damping coefficient' is $\varepsilon = h\omega_0$)

$$\ddot{x}_r(t) + 2\varepsilon\ \dot{x}_r(t) + \omega_0^2 x_r(t) = -\ddot{u}_g(t)$$

Rapid movements ($T < T_0$, $\omega > \omega_0$) of the mass:
acceleration ($\ddot{x}_r$) is high >
the instrument measures ground displacement $u_g.(\ddot{x}_r = \ddot{u}_g \Leftrightarrow x_r \approx u_g)$

Slow movements ($T > T_0$, $\omega < \omega_0$) of the mass:
acceleration ($\ddot{x}_r$) and velocity ($\dot{x}_r$) are low >
the instrument measures the ground acceleration $\ddot{u}_g$. $(x_r \approx \ddot{u}_g)$

⇓

Pendulums are therefore instruments with a resonance frequency much <u>lower</u> than the frequency of the expected seismic signal.

Accelerometers are therefore instruments with an resonance frequency much <u>higher</u> than the frequency of the expected seismic signal.

COMPARISON

Instruments measuring displacement, velocities and accelerations differ in their construction. Considering:

$$\ddot{x}_r(t) + 2h\omega_0\dot{x}_r(t) + \omega_0^2 x_r(t) = -\ddot{u}_g(t)$$

To observe rapid movements of the ground relative to the instrument's eigenperiod (**$\omega_{signal} > \omega_0$, $T_{signal} < T_0$**), accelerations of the mass will be high compared with velocities and corresponding displacements, hence $\dot{x}_r(t), x_r(t)$ will be negligible and

$$\ddot{x}_r(t) = -\ddot{u}_g(t)$$

$$\ddot{x}_r(t) \approx u_g(t)\omega^2$$

$$\frac{\ddot{x}_r(t)}{\omega^2} \approx u_g(t)$$

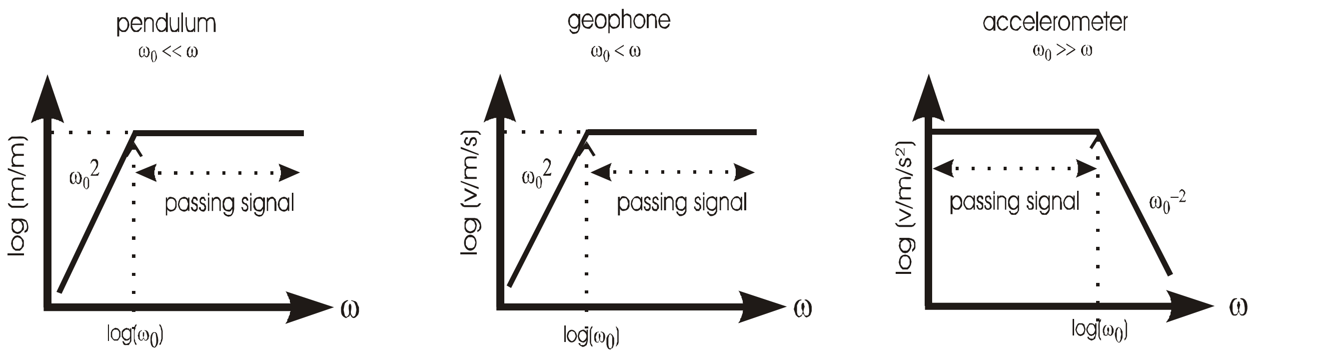
and the sensor measures **ground displacement.** These instruments are likely to be affected by ground tilt, temperature and air pressure effects.

To observe slow movements of the ground relative to the instrument's natural period (**$\omega_{signal} < \omega_0$, $T_{signal} > T_0$**), displacements of the mass will be high compared with velocities and corresponding accelerations, hence $\dot{x}_r(t), \ddot{x}_r(t)$ will be negligible and

$$\omega_0^2 x_r(t) \approx -\ddot{u}_g(t)$$

and the sensor measures **ground acceleration.** Note, that 'x_r' is usually very small which results in a small sensitivity lending itself to be used as a strong ground-motion instrument.

Frequency Response Functions



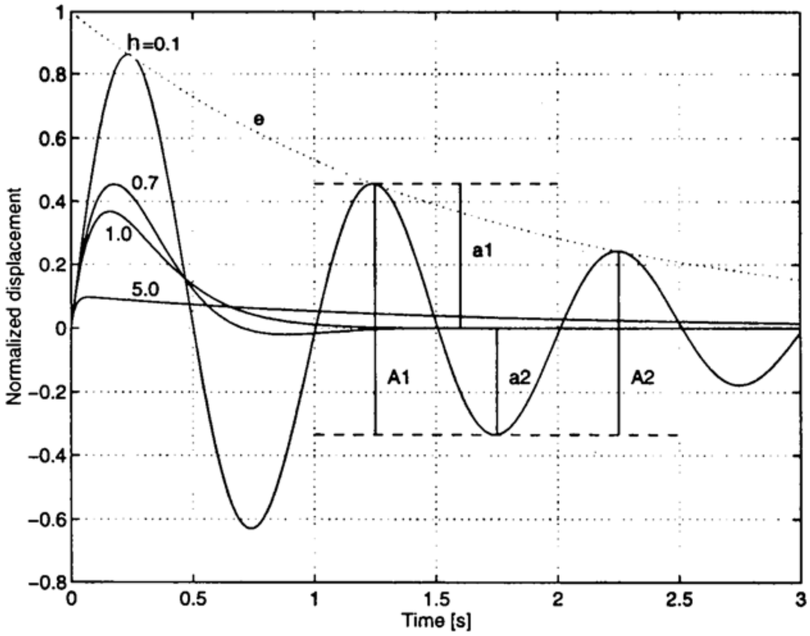
Pendulums are therefore instruments with an eigenfrequency much lower than the frequency of the expected seismic signal.

Geophones are therefore instruments with an eigenfrequency lower than the frequency of the expected seismic signal. Hence, they operate at a most useful bandwidth above the natural frequency and exhibit a relatively narrow usable bandwidth.

Accelerometers are therefore instruments with an eigenfrequency much higher than the frequency of the expected seismic signal.

DAMPING

The damping coefficient ‘ε’ can be determined from the logarithmic decrement '$\Delta_{1/2}$':



m0106

$$\Delta_{1/2} = \ln\left(\frac{a_1}{a_2}\right) = \ln\frac{e^{-\varepsilon t}}{e^{-\varepsilon(t+\pi/\omega)}} = \frac{\pi\,\varepsilon}{\omega} = \frac{\pi\,h}{\sqrt{1-h^2}} = \frac{\varepsilon\,T}{2}$$

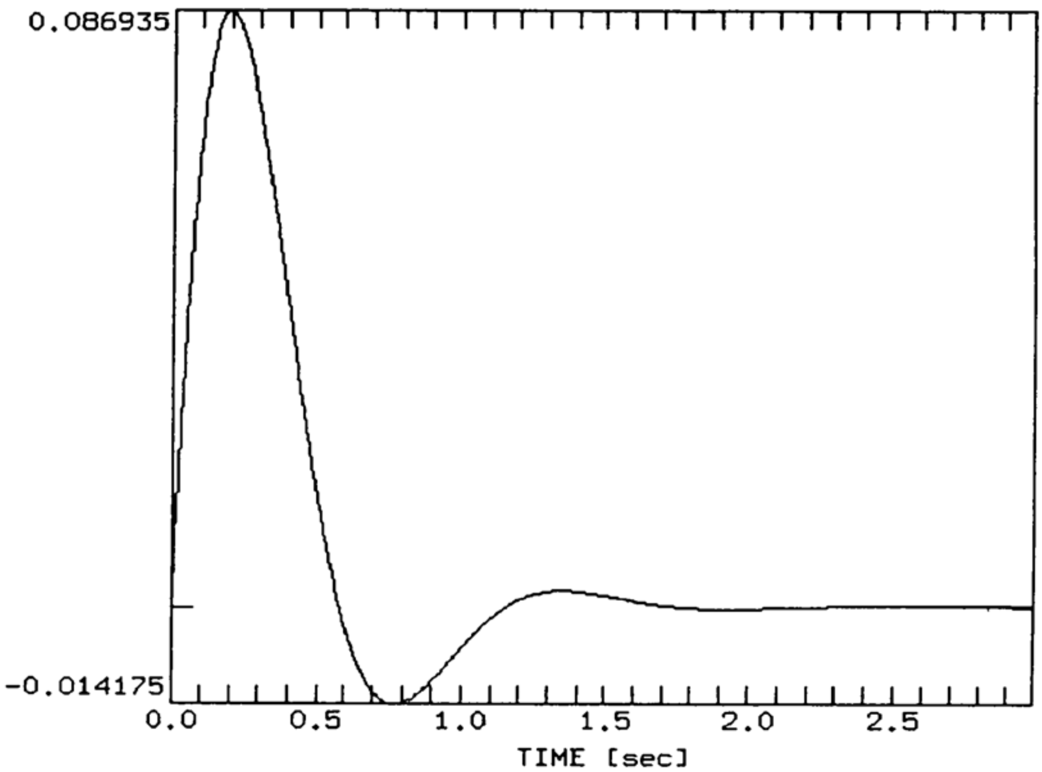
$$h = \frac{\Delta_{1/2}}{\sqrt{\pi^2+\Delta_{1/2}{}^2}} = \frac{\varepsilon}{\omega_0} = \frac{D}{2m\omega_0}$$

whereas a_1 and a_2 are amplitudes of consecutive peaks (1st maximum, 1st minimum)

$\varepsilon = 0$	undamped	resonance
$\varepsilon << \omega_0$ $h < 0.5$	extremely underdamped	ringing
$\varepsilon < \omega_0$ $h < 1$	underdamped	$x_r(t) = \frac{x_{r0}}{\cos\theta} e^{-\varepsilon\,t} \cos(\omega t - \theta)$ $\theta = \arcsin\left(\frac{\varepsilon}{\omega_0}\right)$ oscillates with $T = \frac{T_0}{\sqrt{1-h^2}}$
$\varepsilon = \omega_0$ $h = 1$	critically	$x_r(t) = x_{r0}(\varepsilon\,t+1)e^{-\varepsilon t}$ $T \longrightarrow \infty$
$\varepsilon > \omega_0$ $h > 1$	overdamped	$x_r(t) = A_1 e^{-c_1 t} + A_2 e^{-c_2 t}$ slow restitution, disturbs later arrivals

desired $\varepsilon < \omega_0$ (underdamped case)

CALIBRATION



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The damping constant 'h' and the eigenperiod 'T_0' can be evaluated from the first two amplitude peaks and the time of the second zero-crossing 'T':

a_1 = 0.086935
a_2 = -0.014175
T = 1.1547 sec

$$\textit{hence}$$

$$\left(\frac{|a_1|}{|a_2|} = 6.13297\right) \Rightarrow \Delta_{1/2} = 1.81368$$

$$h = \frac{\Delta_{1/2}}{\sqrt{\pi^2 + \Delta_{1/2}{}^2}} = 0.5$$

$$T_0 = T\sqrt{1-h^2} = 1\,\mathrm{sec}$$

$$\textit{because}$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\omega_0^2 - \varepsilon^2}} = \frac{2\pi}{\omega_0^2\sqrt{1-\frac{\varepsilon^2}{\omega_0^2}}} = \frac{T_0}{\sqrt{1-h^2}}$$

FREQUENCY RESPONSE FUNCTION

A harmonic force

$$\ddot{u}_g(t) = -\omega^2 A_{Input} e^{j\omega t}$$

causes the seismometer to react:

$$x_r(t) = A_{Output} e^{j\omega t}$$

$$\dot{x}_r(t) = j\omega \ A_{Output} e^{j\omega t}$$

$$\ddot{x}_r(t) = -\omega^2 A_{Output} e^{j\omega t}$$

with 'A_{Input}' being the input-displacement,
'A_{Output}' being the displacement of the mass within the seismometer (output-displacement). j = √-1.

Based on

$$\ddot{x}_r(t) + 2\varepsilon \ \dot{x}_r(t) + \omega_0^2 x_r(t) = -\ddot{u}_g(t)$$

we get

$$-\omega^2 A_{Output} + 2\varepsilon \ j\omega \ A_{Output} t + \omega_0^2 A_{Output} = \omega^2 A_{Input}$$

The 'frequency response function' is finally given by the relation of 'Output' to 'Input':

$$\frac{Output}{Input} = \frac{A_o}{A_i} = \frac{\omega^2}{\omega_0^2 - \omega^2 + j2\varepsilon\omega} = T(j\omega)$$

or, in other terms,

$$\left|T(j\omega)\right| = \frac{\omega^2}{\sqrt{(\omega_0{}^2 - \omega^2)^2 + 4\varepsilon^2\omega^2}}$$

$$\phi(\omega) = \arctan\left(\frac{-2\varepsilon\omega}{\omega_0{}^2 - \omega^2}\right)$$

and

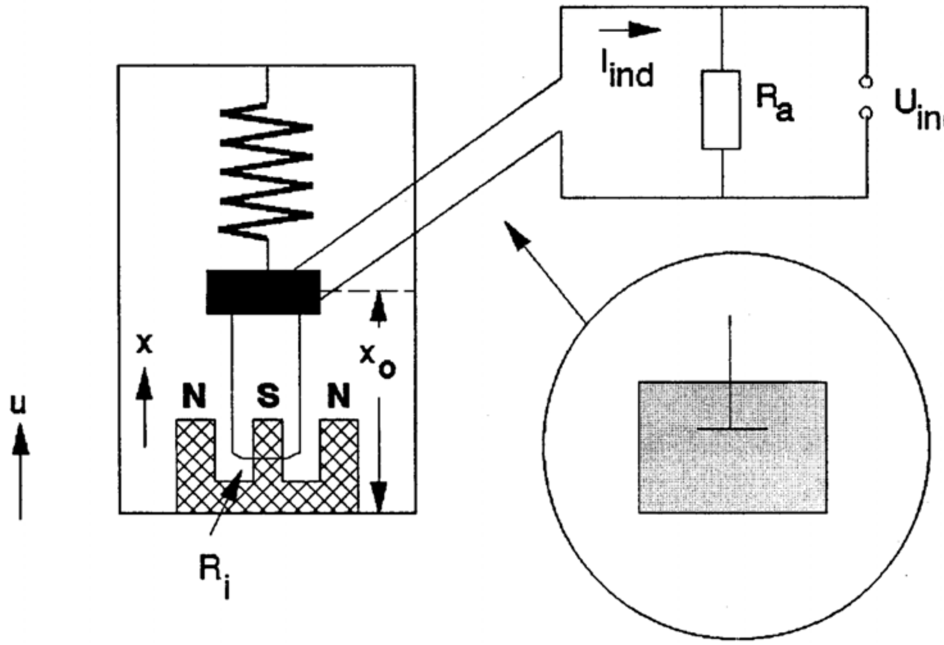
$$T(j\omega) = \left|T(j\omega)\right| e^{j\phi(\omega)}$$

This is the 'frequency response' of a pendulum!

(The pendulum measures displacement at $\omega > \omega_0$ ⇨ rapid ground movement)

Note: The 'frequency response function' can be expressed by the Fourier transform of the outgoing signal divided by the Fourier transform of the incoming signal.

THE ELECTRODYNAMIC SYSTEM



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Dashpot is replaced by coil.

$$I_{induced} = \frac{U_{induced}}{R_a + R_i}$$

R_a... shunt resistance, R_i... internal resistance

$$\varepsilon = \underbrace{\varepsilon_0}_{pendulum(spring)} + \underbrace{\frac{b}{R_a + R_i}}_{coil}$$

hence, the damping constant 'h' of the combined system is (h = ε / ω_0)

$$h = \underbrace{h_0}_{pendulum(spring)} + \underbrace{\frac{b'}{R_a + R_i}}_{coil}$$

with '*b'* = b / ω_0 (= relative damping factor)

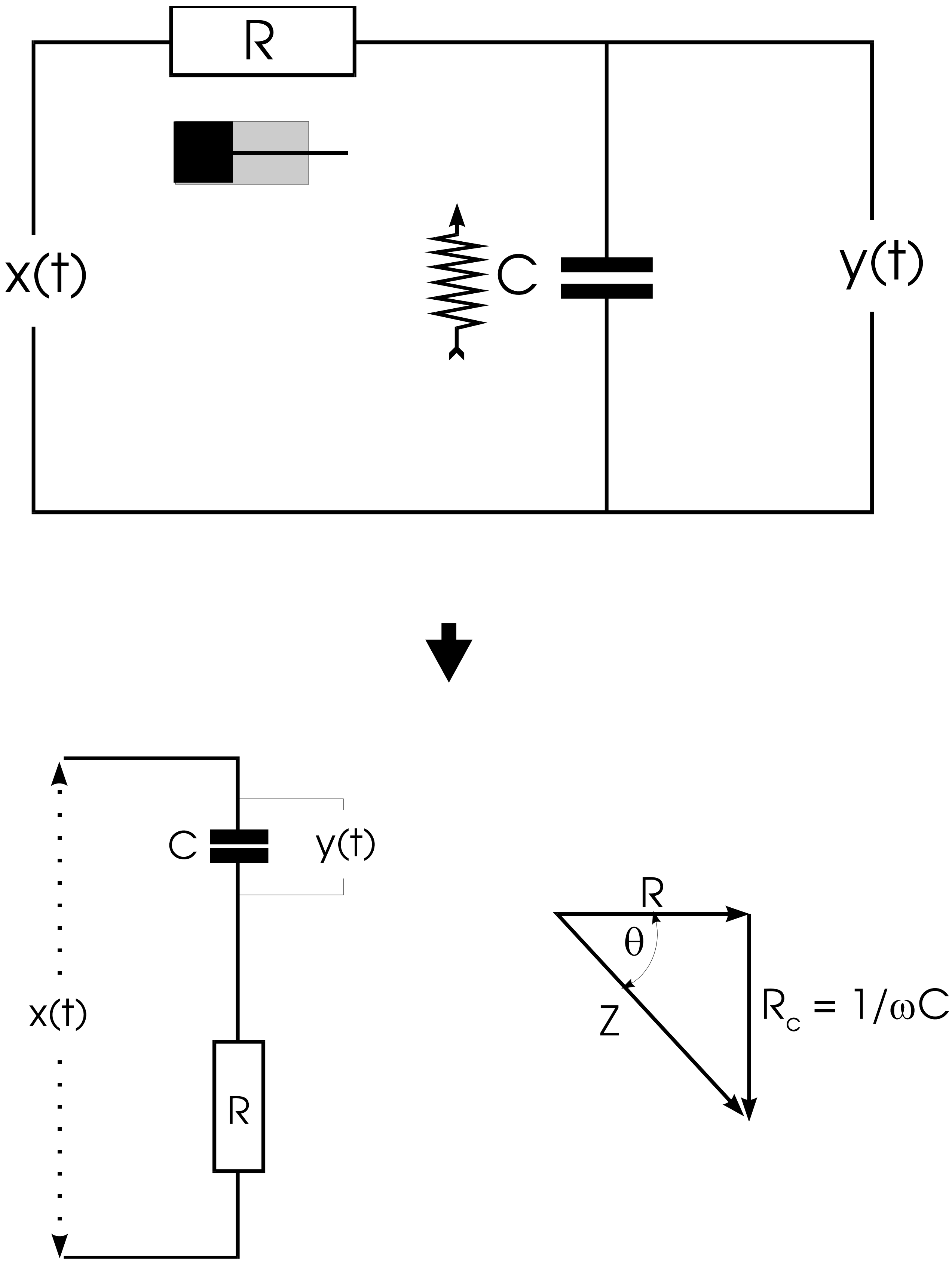
Since $U_{induced} = const. \approx \dot{x}_r(t) = x_r(t)\ \omega$, the 'displacement frequency response' of an electrodynamic system is given by

$$|T(j\omega)| = \omega\, G \frac{\omega^2}{\sqrt{(\omega_0^{\ 2} - \omega^2)^2 + 4\varepsilon^2\omega^2}}$$

with G = generator constant (output voltage/ground velocity) ⇒ [V/m/s]

SYSTEM THEORY

A time-dependent voltage is applied at x(t). The RC-filter consists of a resistor 'R' (produces the damping in the system ⇒ electronic equivalent of the dashpot) and a capacitor 'C' (⇒ electronic equivalent of the spring).



$$\theta = \arccos\left(\frac{R}{Z}\right); Z = \sqrt{R^2 + R_c^2} \longleftarrow R_c = \frac{1}{\omega C}$$

$$I = \frac{U}{Z} \longrightarrow U = IZ; U_c = IR_c; RC = \frac{1}{\omega_0} \Rightarrow$$

$$\frac{U_c}{U} = \frac{IR_c}{IZ} = \frac{R_c}{\sqrt{R^2 + R_c^2}} = \frac{1}{\sqrt{1 + (RC\omega)^2}} = \frac{1}{\sqrt{1 + \frac{\omega^2}{\omega_0^2}}}$$

Or in other words: At 'y(t)' we measure the voltage difference

$$y(t) = x(t) - RI(t)$$

The current 'I(t)' is controlled by the capacitance 'C':

$$I(t) = C\dot{y}(t)$$

hence

$$RC\dot{y}(t) + y(t) - x(t) = 0$$

This is a '<u>first order linear differential equation</u>', which is
1) a linear system (see equation)
2) time invariant (R and C don't change)

For

$$y(t) = A_{Output} e^{j\omega t} \Rightarrow \dot{y}(t) = j\omega A_{Output} e^{\omega t}$$

$$x(t) = A_{Input} e^{j\omega t}$$

we get

$$\frac{A_{Output}}{A_{Input}} = \frac{1}{1 + j\omega RC} = T(j\omega)$$

(one-pole low pass filter with time-constant 'RC')

$$\Downarrow$$

$$T(j\omega) = \frac{1}{\tau}\left[\frac{1}{\frac{1}{\tau} + j\omega}\right] = \frac{1}{\sqrt{1 + \frac{\omega^2}{\omega_c^2}}}$$

TRANSFER FUNCTION

$$\ddot{x}_r(t) + 2\varepsilon\ \dot{x}_r(t) + \omega_0^2 x_r(t) = -\ddot{u}_g(t)$$

$$\Downarrow$$

Laplace Transform ($L\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-st}dt; \cdots s = \sigma + j\omega;$ $j = i$ in electro - technics)

$$s^2 X_r(s) + 2\varepsilon\ sX_r(s) + \omega_0^2 X_r(s) = -s^2 U_g(s)$$

$$\Downarrow$$

$$T_{displ}(s) = \frac{X_r(s)}{U_g(s)} = \frac{-s^2}{s^2 + 2\varepsilon\ s + \omega_0^2}$$

or electrodynamic

$$T_{vel}(s) = G_{vel}\frac{-s^2}{s^2 + 2\varepsilon\ s + \omega_0^2} \longleftrightarrow T_{displ}(s) = G_{displ}\frac{-s^3}{s^2 + 2\varepsilon\ s + \omega_0^2}$$

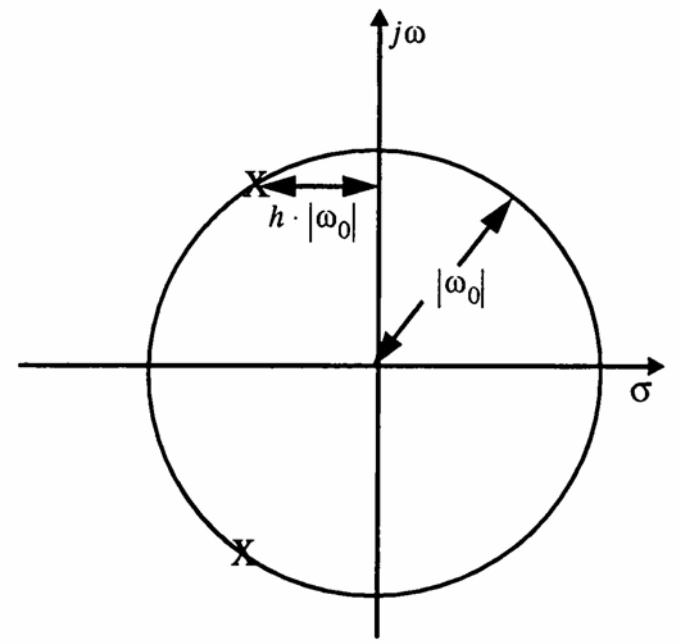
The roots in the denominator (poles) are

$$p_{1,2} = -\varepsilon \pm \sqrt{\varepsilon^2 - \omega_0^2} = -\left(h \pm \sqrt{h^2 - 1}\right)\omega_0$$

and in the underdamped (h<1) case:

$$p_{1,2} = -\left(h \pm j\sqrt{1 - h^2}\right)\omega_0$$

$$\left|p_{1,2}\right| = \left|\omega_0\right|$$



s0405

Pole position 'X', resonance frequency 'ω_0' and damping 'h' for a seismometer in the s-plane.

FREQUENCY RESPONSE vs. TRANSFER FUNCTION

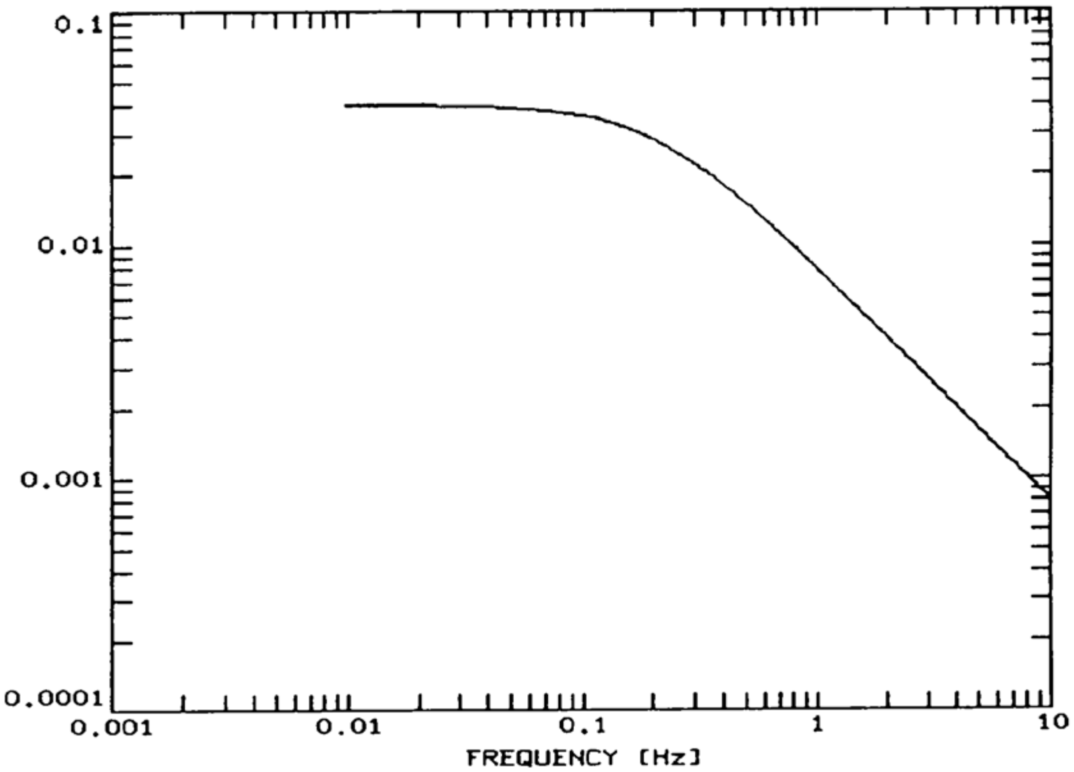
Frequency response	Transfer function
applies to	
applies to stationary ground oscillations	transient ground motions
The function is defined as	
$T(j\omega) = Y(j\omega) / X(j\omega)$	$T(s) = Y(s) / X(s)$
and can be generally described by	
no general definition	poles & zeros
The advantages are:	
1) easy to calculate and 2) used in 'existing systems' for considering the system response	1) used to design system performances 2) the 'physical concept' is explicitly known
Can be achieved by	
⇓ Fourier transform	⇓ Laplace transform

POLES AND ZEROS

The transfer function T(s) is special case of the frequency response

$$|T(j\omega)| = \frac{1}{\sqrt{1+\frac{\omega^2}{\omega_0^2}}}$$

which can be expressed in a log-log fashion:



s0211

The frequency response decreases for frequencies above the eigenfrequency ω_0 (example shows 0.2 Hz) with 20dB/decade (= 1:1)

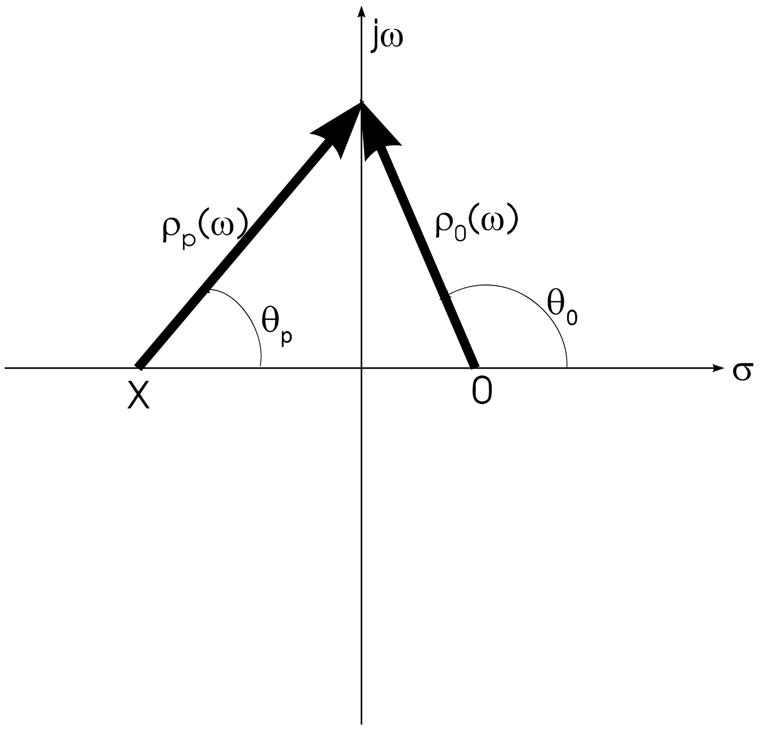
$$\left|\frac{-1}{\tau}\right| = |\omega_0| = \frac{1}{RC}$$

which is called a pole in the s-plane.

The inverse function leads to a zero instead of a pole thus causing the frequency response to increase above ω_0.

The frequency-response function of a RC-filter is completely defined by one pole and the inverse frequency-response function is defined by one zero on the real axis of the s-plane.

Graphical representation of a system having one pole (X) and one zero (0):



The transfer function of this system becomes (proof see under LTI-systems)

$$T(s) = \frac{s - s_0}{s - s_p}$$

Hence, the frequency response function is

$$T(j\omega) = \frac{j\omega - s_0}{j\omega - s_p}$$

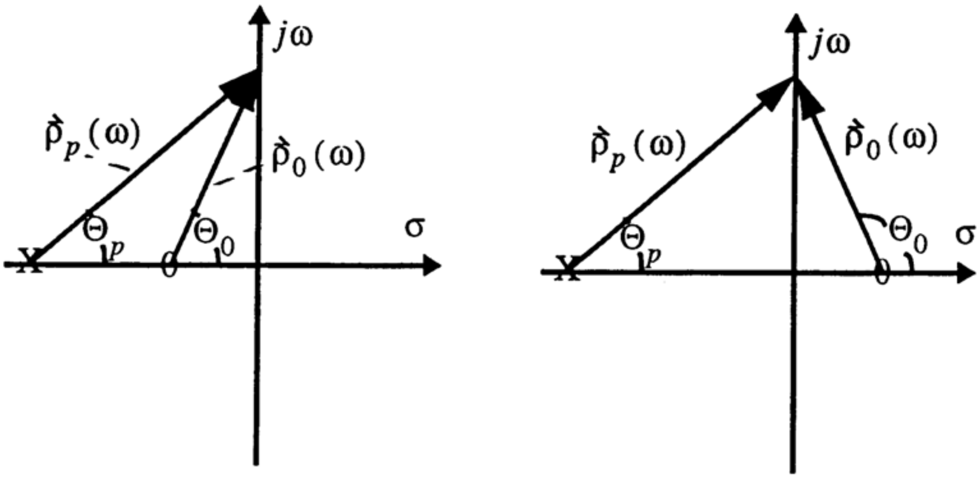
or

$$T(j\omega) = \left|\vec{\rho}_0(\omega)\right| e^{j\theta_0} \frac{1}{\left|\vec{\rho}_p(\omega)\right|} e^{-j\theta_p} = \frac{\left|\vec{\rho}_0(\omega)\right|}{\left|\vec{\rho}_p(\omega)\right|} e^{j(\theta_0 - \theta_p)}$$

The product of vectors pointing from the zeros to 'jω' is divided by the product of vectors pointing from the poles to 'jω' to arrive at the frequency dependent amplitude response.

The sum of phases of poles are subtracted from the sum of phases of zeros to arrive at the frequency dependent phase response.

PHASE PROPERTIES



s0303

minimum phase   maximum phase

The complex s-plane representation of stable 'one pole/one zero'-systems, having identical amplitude- but different phase response.

filter	comment
minimum phase	no zeros in the right half plane
maximum phase	all zeros in the right half plane
mixed phase	between minimum- and maximum phase
linear phase	no phase distortion, but constant shift at all frequencies $x(t-a) \Leftrightarrow X(j\omega)e^{-j\omega a}; a > 0$
zero phase	phase response zero for all frequencies (filtering twice in opposite direction, no real-time processing possible!)
all pass	amplitude remains constant, phase response changes

A causal stable system has no poles in the right half of the s-plane!

LTI-SYSTEM

(Linear Time Invariant System)

The differential equation of an electric circuit (RC filter):

$$RC\dot{y}(t) + y(t) - x(t) = \alpha_1 \frac{d}{dt} y(t) + \alpha_0 y(t) + \beta_0 x(t) = 0$$

is a special case (1st order system) of an n-th order LTI-system:

$$\sum_{k=0}^{n} \alpha_k \frac{d^k}{dt} y(t) + \sum_{k=0}^{m} \beta_k \frac{d^k}{dt} x(t) = 0$$

The transfer function of an n-th order system is

$$T(s) = \frac{-\sum_{k=0}^{m} \beta_k s^k}{\sum_{k=0}^{n} \alpha_k s^k} = \frac{-\beta_m \prod_{k=1}^{m} (s - s_{0k})}{\alpha_n \prod_{k=1}^{n} (s - s_{pk})}$$

Hence, the transfer function of a RC-filter is given by

$$T(s) = \frac{Y(s)}{X(s)} = \frac{-\beta_0}{\alpha_0 + \alpha_1 s}$$

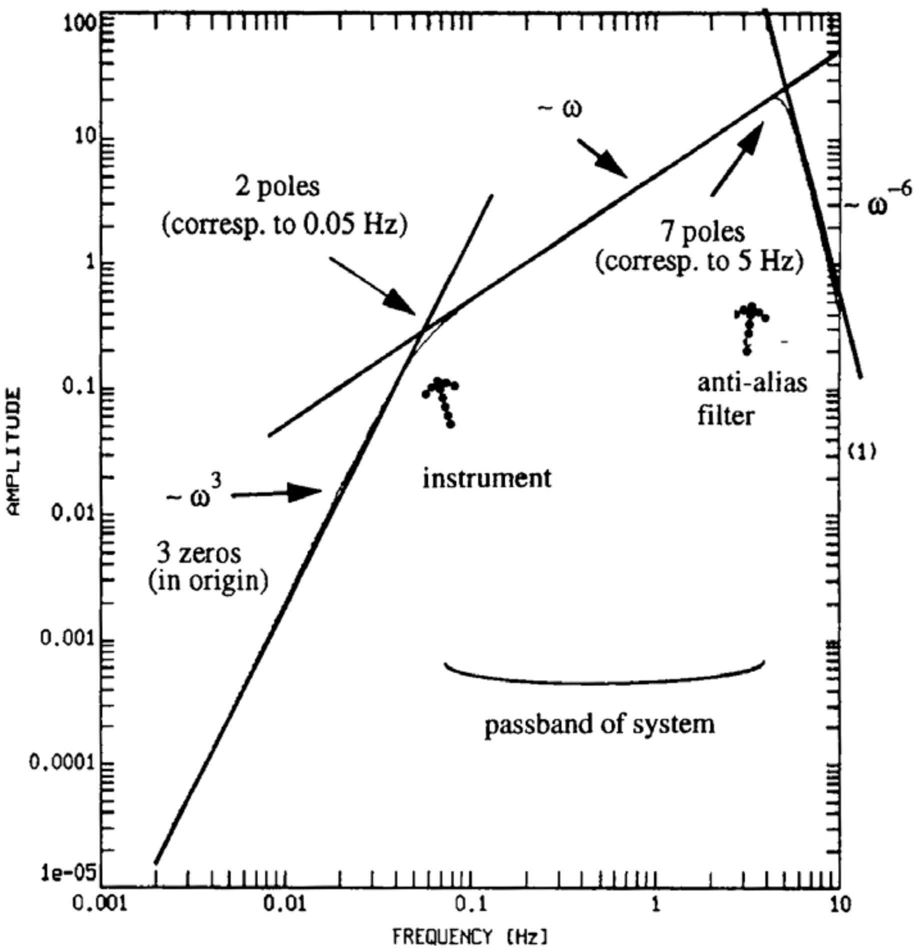
In terms of poles and zeros we may express the transfer function as

$$T(s) = \frac{-\beta_0}{\alpha_1 (s - s_{p1})}$$

For an RC-filter, $\beta_0 = -1$, $\alpha_0 = 1$ and $\alpha_1 = RC$,
the filter has no zeros, but a single pole at

$$s_p = \frac{-1}{\alpha_1} = \frac{-1}{\tau} = \frac{-1}{RC}$$

DETERMINING POLES AND ZEROS



Sa_0309

Frequency response of an 'unknown' pole-zero distribution (see also Scherbaum, F. 1996).

Procedure:

1. determine slopes
2. determine 'corner-frequencies'
3. define number of poles and zeros

Example:

(see figure)

1. slopes are ω^3, ω, ω^{-6}
2. corner frequencies are at: $\omega^3 \leftrightarrow \omega$ (0.05 Hz) and $\omega \leftrightarrow \omega^{-6}$ (5 Hz)
3. zeros: 3 zeros ($\equiv \omega^3$) at origin (frequency = 0 Hz)
   poles: 2 poles ($\equiv \omega^3 \leftrightarrow \omega$) at 0.05 Hz
   7 poles ($\equiv \omega \leftrightarrow \omega^{-6}$) at 5 Hz

CALIBRATION FILE

A sensor with the following characteristics is given:

- The sensor generates a voltage above 1 Hz ($\omega_0 = 6.283$) is proportional to ground velocity
- Damping 'h' = 0.7
- The generator constant 'G' = 100 V/m/s, the signal amplification before A/D conversion = 250 and the least significant bit of the A/D-conversion (LSB) for converting Volts into digital counts is 1μV, or in other words 1V = 10^6 counts.

Transfer Functions

Velocity Transfer Function

$$T_{vel}(s) = -100\left[\frac{V}{m/s}\right]\frac{s^2}{s^2 + 8.7964s + 39.476}$$

Displacement Transfer Function
is given by multiplying the velocity transfer function by 's'

$$T_{disp}(s) = -100\left[\frac{V}{m}\right]\frac{s^3}{s^2 + 8.7964s + 39.476}$$

Poles & Zeros

Poles

$$s_{p(1,2)} = -(h \pm \sqrt{h^2 - 1})\omega_0 \longrightarrow s_{p(1,2)} = -(0.7 \pm j0.71414)6.283$$

hence, we arrive at two poles

$$s_{p(1)} = -(4.398 + j4.487)$$

$$s_{p(2)} = -(4.398 - j4.487)$$

Zeros

$s^3 \Rightarrow$ 3 zeros at the origin of s-plane

GSE Format

For establishing a proper calibration file in the GSE (Global Scientific Experts) format, the generator constant 'G' (100 V/m/s) needs to be multiplied by the pre-amplifier constant of 250, we get 2.5 10^4 V/m. This value has to be multiplied again by 10^6 to take account of the LSB and divided by 10^9 to convert the constant to counts/nm to comply with the GSE format. Therefore, a calibration file in the GSE format would look like:

```
CAL1 1Hz                      PAZ
2
-4.398  4.487
-4.398 -4.487
3
0.0000  0.000
0.0000  0.000
0.0000  0.000
25.0
```

S-PLANE ⇔ Z-PLANE

Purpose: Representation of discrete time series

$L \quad \Rightarrow \quad z$

continuous **discrete**

Principle:

$$z = e^{sT} = e^{(\sigma + j\omega)T} = e^{\sigma T} e^{j\omega T} = re^{j\omega T}$$

$$L\{x(t)\delta_T(t)\} = \int_{-\infty}^{\infty} x(t)\delta_T(t) e^{-st} dt =$$

$$\int_{-\infty}^{\infty} x(t) \left(\sum_{n=-\infty}^{\infty} \delta\ (t - nT) \right) e^{-st} dt = \sum_{n=-\infty}^{\infty} x(nT) e^{-snT}$$

Note: - ∞ and ∞ similar to the double sided Fourier Transform.

For switching from continuous to discontinuous (discrete) time series, we formally alter

$x(nT) \Rightarrow x[nT]$

and it follows

$$L\{x[nT]\} = \sum_{n=-\infty}^{\infty} x[nT] e^{-snT}$$

with $x[nT]$ = discrete time series with sample interval T

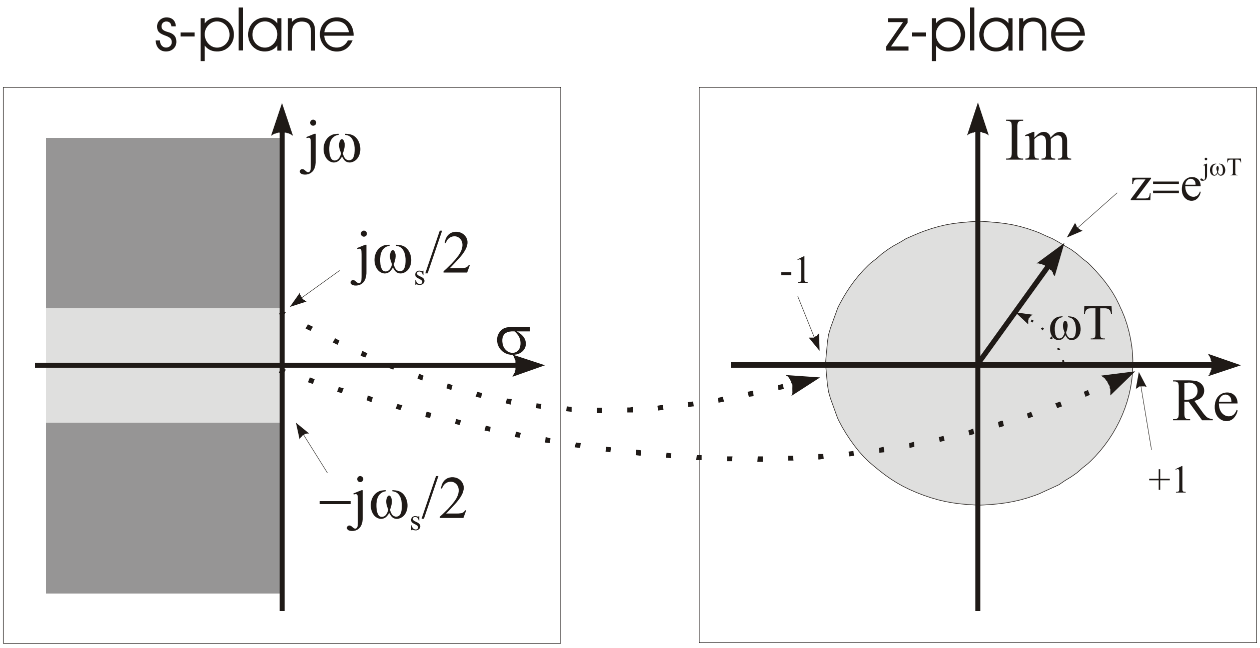
Defining $z = e^{st}$ and $x[n] = x[nT]$, we get

$$Z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = X(z)$$

with z being the continuous complex variable

The z-transfer function is then given by

$$T(z) = \frac{Z\{y[n]\}}{Z\{x[n]\}}$$



T = sampling interval, ω = 2πf is the angular frequency,
ω_s = sampling frequency = 2 * Nyquist frequency

case in s-plane	**position in z-plane**
s = 0	z = 1 (unit circle)
σ < 0 (left side)	r < 1 (inside unit circle)
s = jω	r = 1 (on unit circle) ⇨ Fourier transform
ω > 0	upper half
ω < 0	lower half

⇓ ⇓

all poles on left side	all poles inside unit circle (= causal and stable)
no zeros on right side	no zeros outside unit circle (= minimum phase)

FOURIER ⇒ LAPLACE ⇒ Z

<table>
<tr><th>FOURIER</th><th>LAPLACE</th><th>Z</th></tr>
<tr>
<td>assumes
periodic continuous time series
(harmonic)

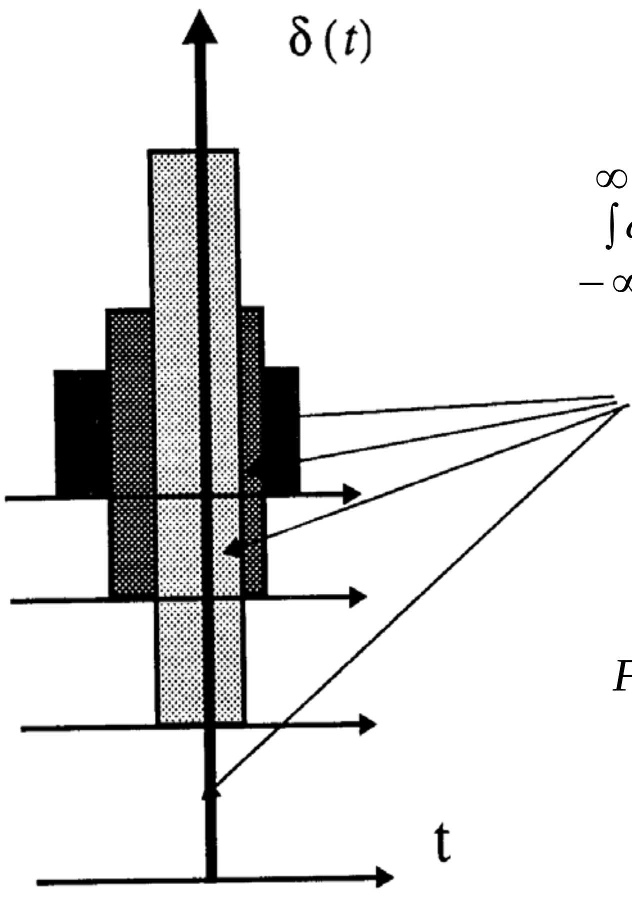
$X(j\omega) = F\{x(t)\} =$
$\int_{-\infty}^{\infty} x(t)e^{j\omega t}dt$</td>
<td>assumes
continuous time series
with exponential decay

$X(s) = L\{x(t)\} =$
$\int_{-\infty}^{\infty} x(t)e^{st}dt$</td>
<td>assumes
discrete time series

$X(z) = Z\{x[t]\} =$
$\sum_{n=-\infty}^{\infty} x[n]z^{-n}$</td>
</tr>
<tr><td colspan="3">Integration</td></tr>
<tr>
<td>$\int_{-\infty}^{t} x(\tau)d\tau \Rightarrow \frac{1}{j\omega}X(j\omega)$</td>
<td>$\int_{-\infty}^{t} x(\tau)d\tau \Rightarrow \frac{1}{s}X(s)$</td>
<td>$\sum_{k=0}^{n-1} x[n] \Rightarrow X(z-1) = \frac{1}{z-1}X(z)$</td>
</tr>
<tr><td colspan="3">Derivative</td></tr>
<tr>
<td>$\frac{d}{dt}x(t) \Rightarrow j\omega X(j\omega)$</td>
<td>$\frac{d}{dt}x(t) \Rightarrow sX(s)$</td>
<td>$x[n] - x[n-1] \Rightarrow (z-1)X(z)$</td>
</tr>
<tr><td colspan="3">Convolution</td></tr>
<tr>
<td>$x(t)*h(t) \Rightarrow X(j\omega)H(j\omega)$</td>
<td>$x(t)*h(t) \Rightarrow X(s)H(s)$</td>
<td>$x_1[n]*x_2[n] \Rightarrow$
$\sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m]$</td>
</tr>
<tr><td colspan="3">Time shift</td></tr>
<tr>
<td>$x(t-a) \Rightarrow e^{-j\omega a}X(j\omega)$</td>
<td>$x(t-a) \Rightarrow e^{-sa}X(s)$</td>
<td>$x[n-n_0] \Rightarrow z^{-n_0}X(z)$
special case (inversion of signal)
$x[-n] \Rightarrow X\left(\frac{1}{z}\right)$</td>
</tr>
</table>

IMPULSE & STEP RESPONSE

Properties of the 'impulse' - or Dirac 'delta' - function δ (t):



s0206

$$\int_{-\infty}^{\infty} \delta(t)dt = 1 \xrightarrow{\text{and}} \delta(t) = 0; t \neq 0$$

$$\text{unit area } = \int_{-\infty}^{t} \delta(t)dt = x(t)$$

$$\delta(t) = \frac{dx(t)}{dt}$$

$$F\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)e^{-j2\pi ft}dt = 1$$

$$L\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)e^{-st}dt = 1$$

A Fourier transform and a Laplace transform of the delta function are '1'.

The frequency response function T(jω) is the **Fourier transform** of the impulse response function h(t).

The transfer function T(s) is the **Laplace transform** of the impulse response function h(t).

$$T(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{Y(j\omega)}{1} \quad \text{for } x(t) = \delta(t)$$

$$T(s) = \frac{Y(s)}{X(s)} = \frac{Y(s)}{1} \quad \text{for } x(t) = \delta(t)$$

The step response is the output signal of a unit-step input signal x(t). The step-response is mainly used for calibration purposes (power off/power on).

type of signal		Laplace transform X(s)
Dirac-impulse	δ(t)	1
unit step	x(t)	1/s because x(t) =∫ δ (t) dt, and the Laplace transform of an integral = X(s)/s

Response to unit step:

$$T(s) = \frac{Y(s)}{X(s)} = \frac{Y(s)}{\frac{1}{s}} = sY(s)$$

The step response function a(t) and the impulse response function h(t) are equivalent descriptions of a system. They are linked to each other by integration or differentiation, respectively.

$$a(t) = \int_{-\infty}^{t} h(\tau) d\tau$$

$$h(t) = \frac{d}{dt} a(t)$$

COMMON FILTER OPERATORS

CHEBYSHEV

$f_S = 4f_C$

with f_s... sample frequency, f_c... cut-off frequency

The filter leads to considerable group delays near the cut-off frequency (problem for broadband systems).

Nth-order Chebyshev polynomial:

$$T_{n+1}(x) = 2x\, T_n(x) - T_{n-1}(x) \quad ; n = 0, 1, ...$$

BUTTERWORTH

Exhibits group delays too, but not as 'sharp' (in terms of amplitude response) as Chebyshev.

A second order Butterworth high cut filter:

$$F(z) = a_0 \frac{1 + 2z^{-1} + z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

BESSEL

$f_S = 8f_C$

Constant group delay (linear phase), peak amplitudes are accurate, little ringing and overshoot due to gentle amplitude response.

Nth-order Bessel polynomial:

$$B_n(x) = (2n-1)B_{n-1}(x) + f^2 B_{n-2}(x)$$

FIR

(Finite Impulse Response, non-recursive)

symmetric, always stable, many coefficients needed for steep filters (slow), realization of specifications easy (linear or zero phase can be defined), transfer function completely defined by zeros

IIR

(Infinite Impulse Response due to recursive filter)

potentially unstable, few coefficients needed for steep filters, difficult (if not impossible) to design for specific characteristics, defined by poles and zeros, phase always distorted within passband of filter.

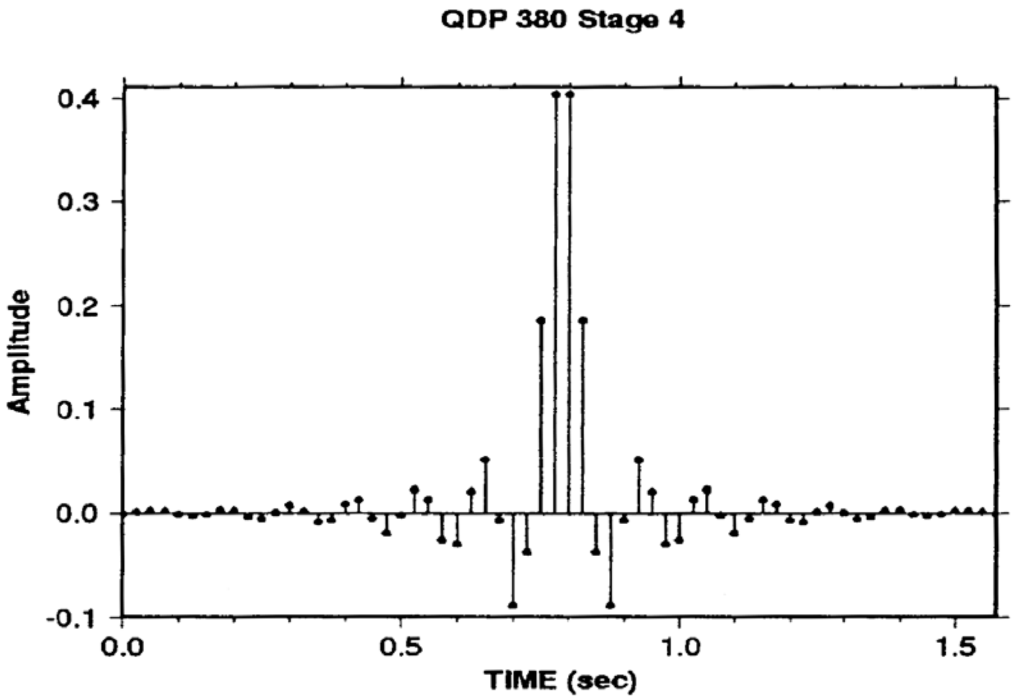
$$x[n-k] \Leftrightarrow z^{-k} X(z)$$

and

$$\sum_{k=0}^{n} a_k y[n-k] = \sum_{k=0}^{m} b_k x[n-k]$$

$$\Downarrow$$

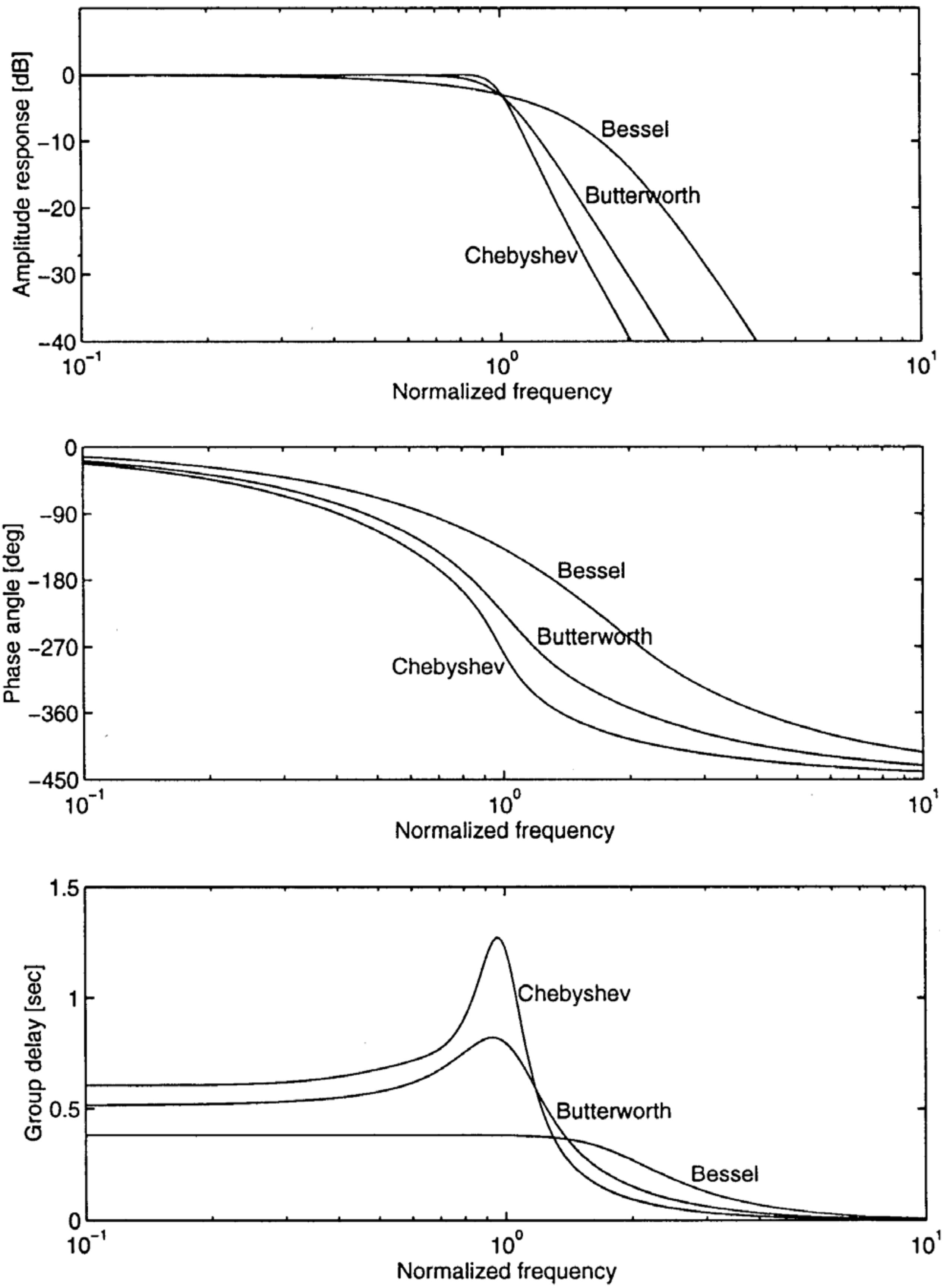
$$T(z) = \frac{\sum_{k=0}^{m} b_k z^{-k}}{\sum_{k=0}^{n} a_k z^{-k}} = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{m} (1 - c_k z^{-1})}{\prod_{k=1}^{n} (1 - d_k z^{-1})}$$



s0801

FIR filter impulse response of the stage 4 of the QDP 380 digitiser by Quanterra causes 'acausal' oscillation (close to the corner frequency of the filter). This effect inhibits exact first onset picking!

COMPARING FILTERS

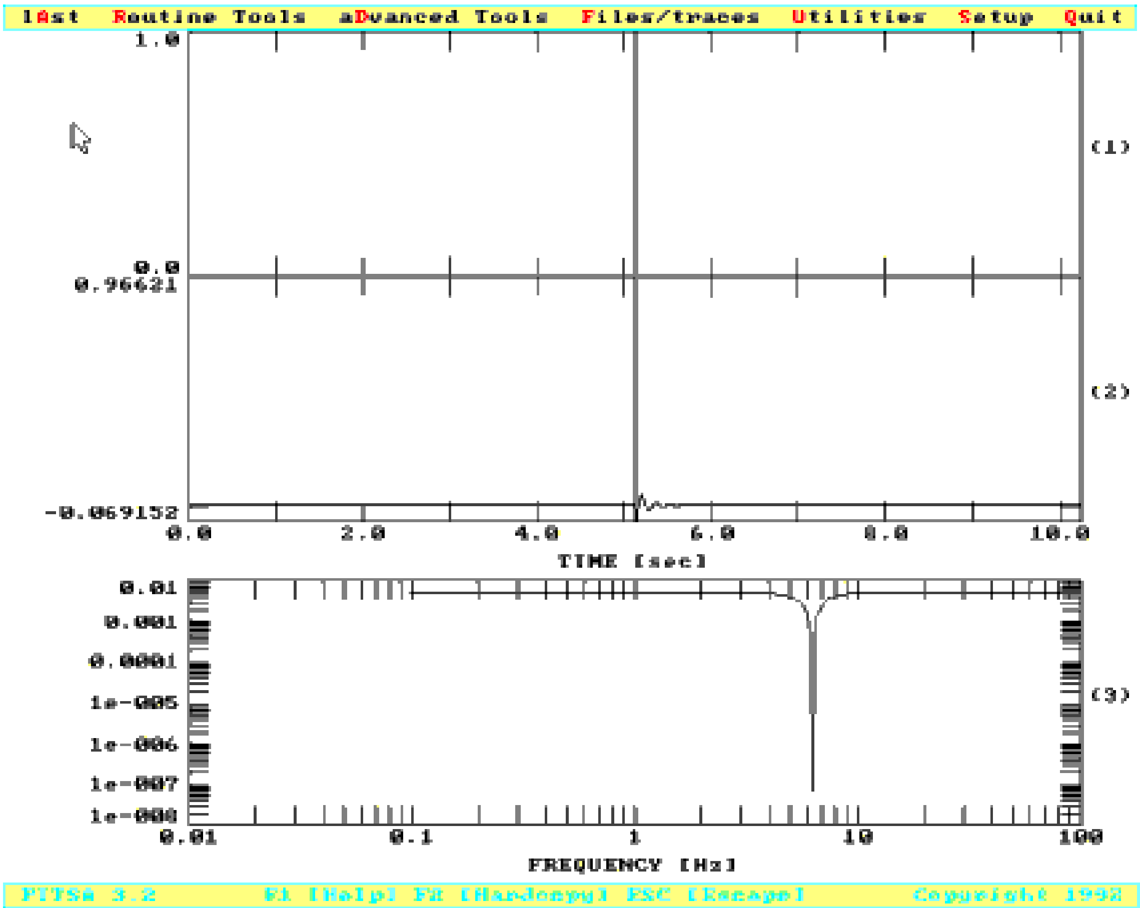


m0202

NOTCH FILTER

Designing filters to eliminate a certain frequency from the recorded spectrum - e.g. 16 2/3 Hz - constitutes a special task. Requirements are :

1. steepness of the filter
2. effectiveness
3. phase should be undistorted



Spike (top trace), impulse response due to poles and zeros (centre) and amplitude response (bottom) for a notch filter eliminating signals near 6.25 Hz . Note: Poles are placed near 'zeros'.

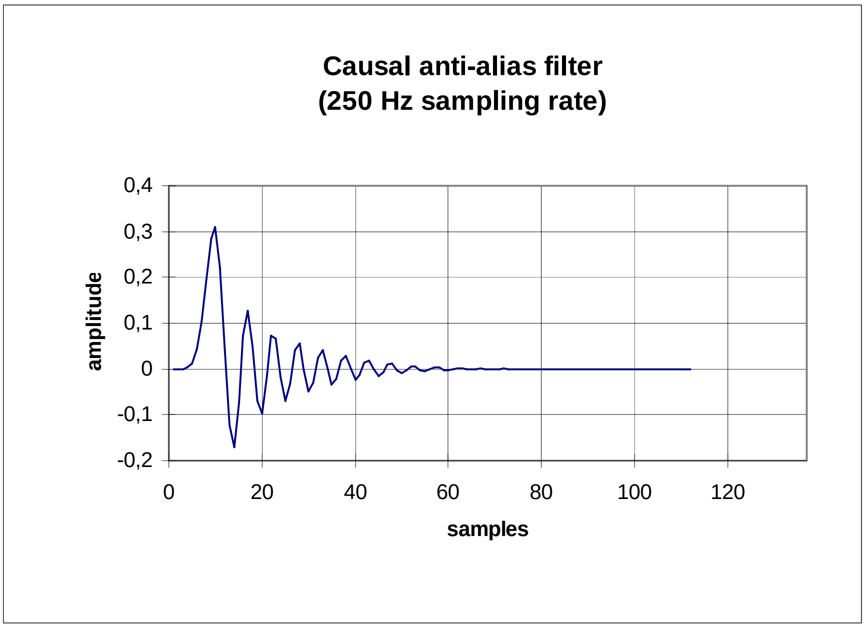
GSE (Global Scientific Experts)-format as required in PITSA:

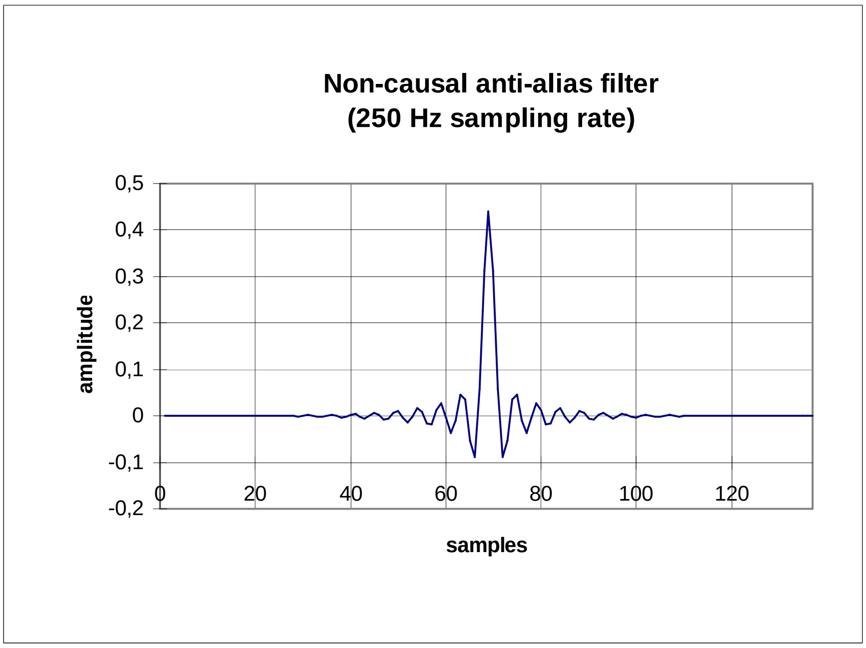
```
CAL1  notch at 6.25Hz          PAZ
2
-6.846 38.828
-6.846 -38.828
2
0.0 39.27
0.0 -39.27
1.0e9
```

CAUSALITY

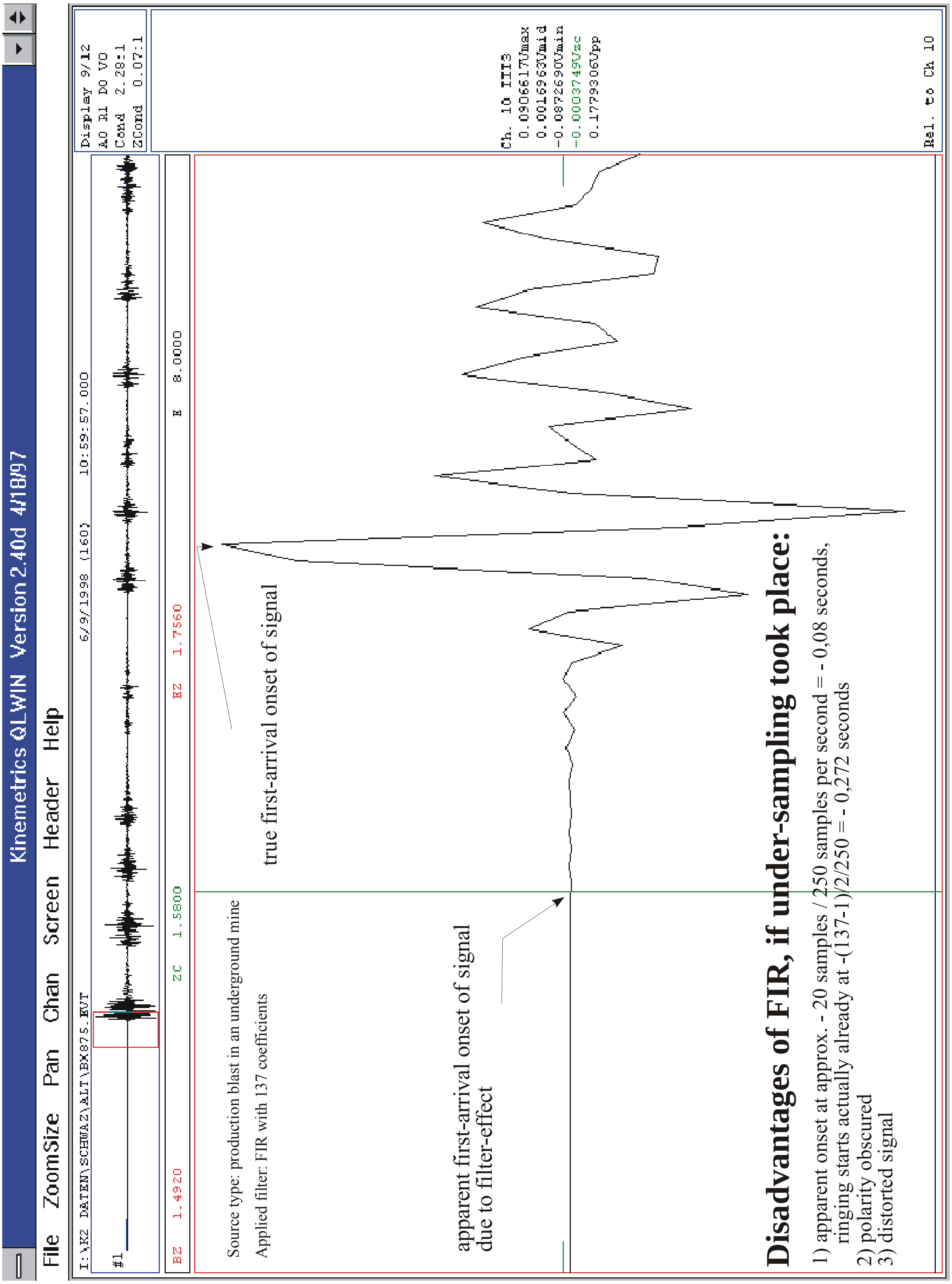
We distinguish between

	causal-filters	non-causal filters
characteristic	asymmetric	symmetric
advantage	can be applied real time	phase information remains
disadvantage	phase distorted	large time-shift, precursor ringing
used for	picking onsets	amplitude, polarization, etc.



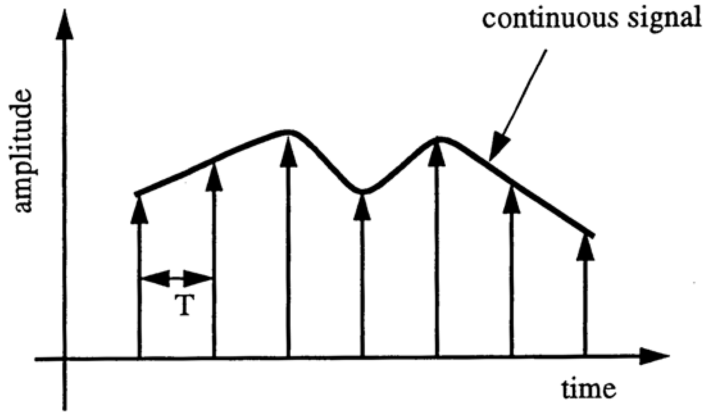


FIR-EFFECT


Kinemetrics QLWIN Version 2.40d 4/18/97
File ZoomSize Pan Chan Screen Header Help
I:\K2 DATEN\SCHWAZ\ALT\BX875.EVT
6/9/1998 (160) 10:59:57.000
#1
Display 9/12
A0 R1 D0 V0
Cond 2.28:1
ZCond 0.07:1
BZ 1.4920
ZC 1.5800
BZ 1.7560
E 8.0000
Ch. 10 III3
0.0906617Vmax
0.0016963Vmid
-0.0872690Vmin
-0.0003749Vzc
0.1779306Vpp
Rel. to Ch 10
Source type: production blast in an underground mine
Applied filter: FIR with 137 coefficients
true first-arrival onset of signal
apparent first-arrival onset of signal due to filter-effect
Disadvantages of FIR, if under-sampling took place:
1) apparent onset at approx. - 20 samples / 250 samples per second = - 0,08 seconds,
ringing starts actually already at -(137-1)/2/250 = - 0,272 seconds
2) polarity obscured
3) distorted signal

SAMPLING

Sampling is the process of taking discrete samples of a continuous data stream.



s0501

The sampling theorem:

For a continuous time signal to be uniquely represented by samples taken at a sampling frequency of *fdig*, (every 1/*fdig* time interval), no energy must be present in the signal at and above the frequency *fdig*/2. *fdig*/2 is commonly called the *Nyquist*[1] *frequency* (sometimes referred to as *folding frequency*). Signal components with energy above the Nyquist frequency will be mapped by the sampling process onto the so-called *alias*-frequencies *falias* within the frequency band of 0 to Nyquist frequency. This effect is called *alias* effect.

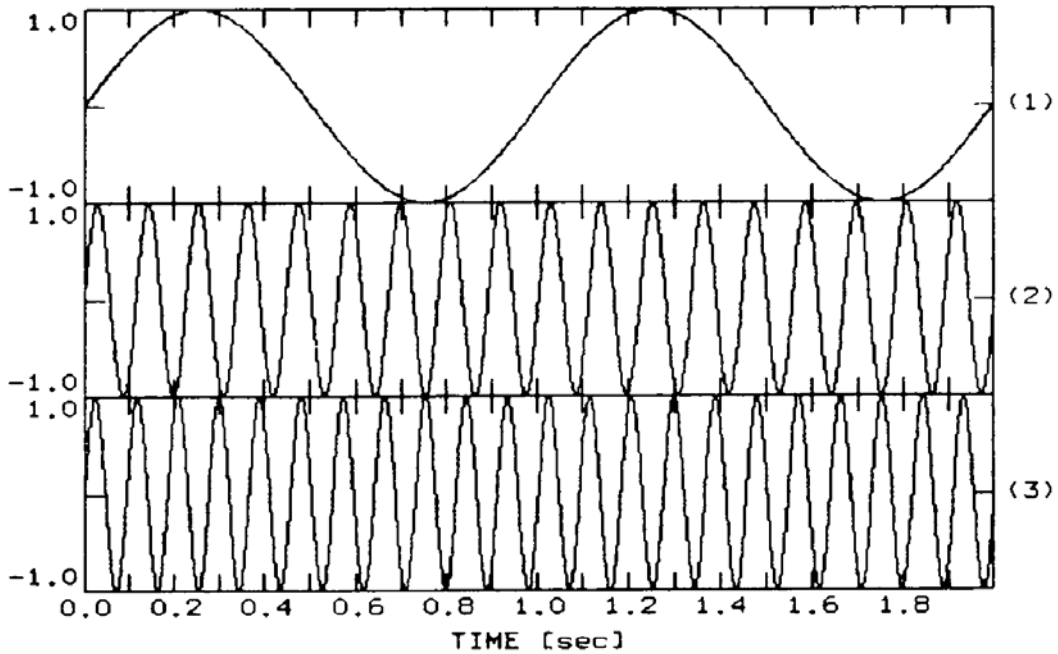
$$f_{alias} = \left| f - n f_{dig} \right| \; ; n \in \Im$$

Example: 'f_{dig}' = 100 Hz, '$f_{Nyquist}$' = 50 Hz

frequency	alias frequency (n=1)	alias frequency (n=2)
60	**40**	140
80	**20**	80
120	**20**	80
140	**40**	60
150	**50**	**50**
180	80	**20**
190	90	**10**

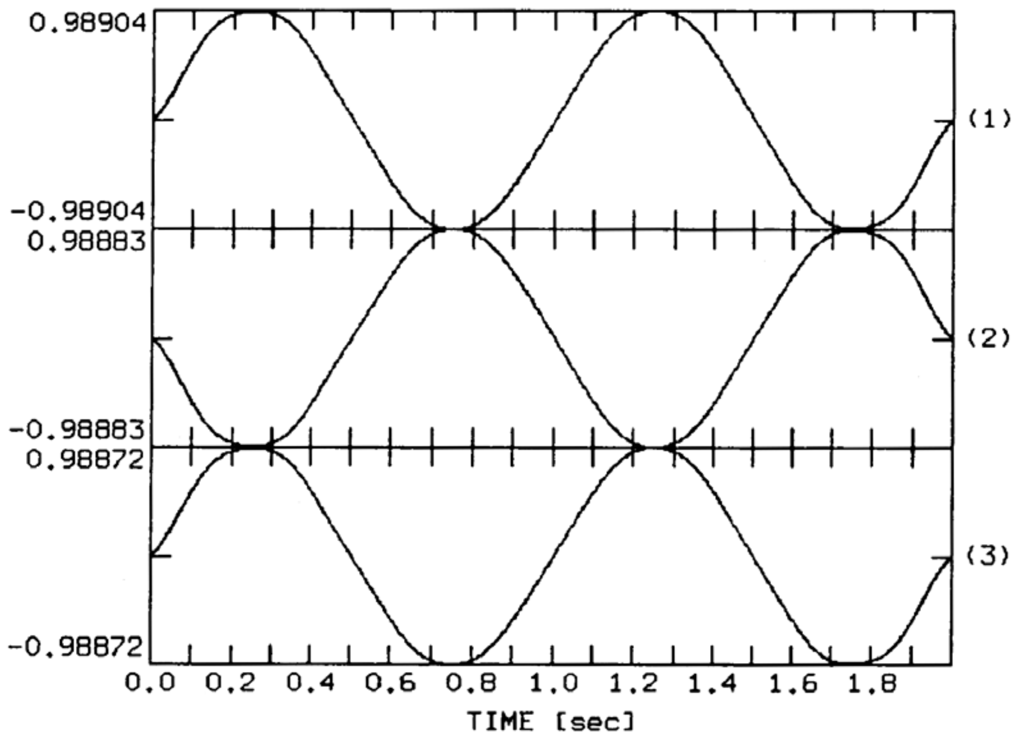
[1]Nyquist, H. (1932). Regeneration Theory. Bell Syst.Techn.Journal, page 126-147.

PROBLEMS WITH SAMPLING



s0502

Input signal with 1, 9, 11 Hz harmonic signals.

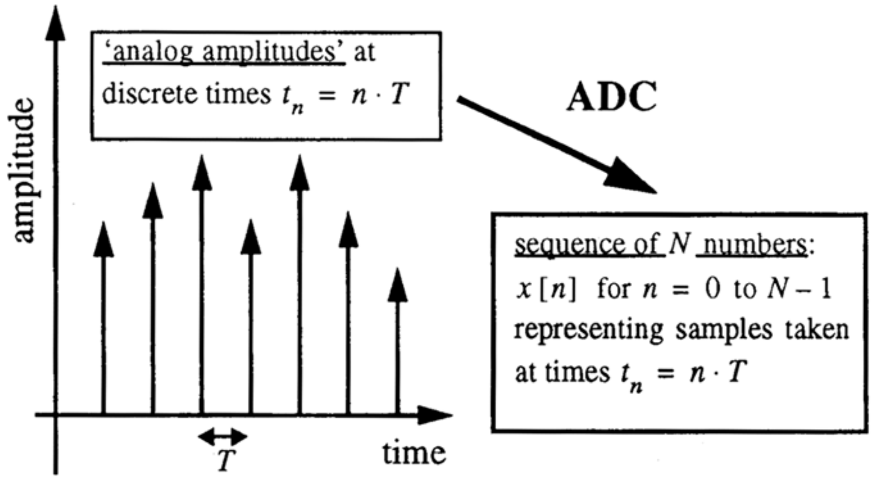


s0502

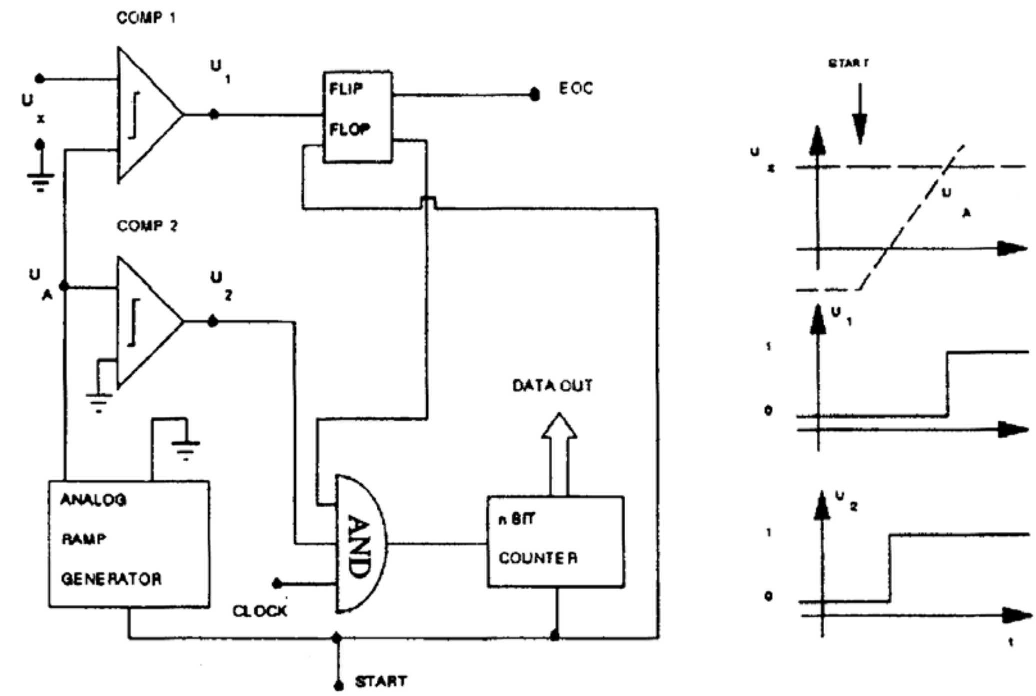
Reconstructed traces after discretizing with 10 Hz sampling frequency.
Note phase shift of second trace!
(from Scherbaum, F. 1996)

ANALOG TO DIGITAL CONVERSION

s0601



Example
(simple and easy to implement, but only working up to 1 kHz)



s0602

Principle: The time it takes 'Ua' to exceed 'Ux' is measured. Each time step is counted and expressed in bits.

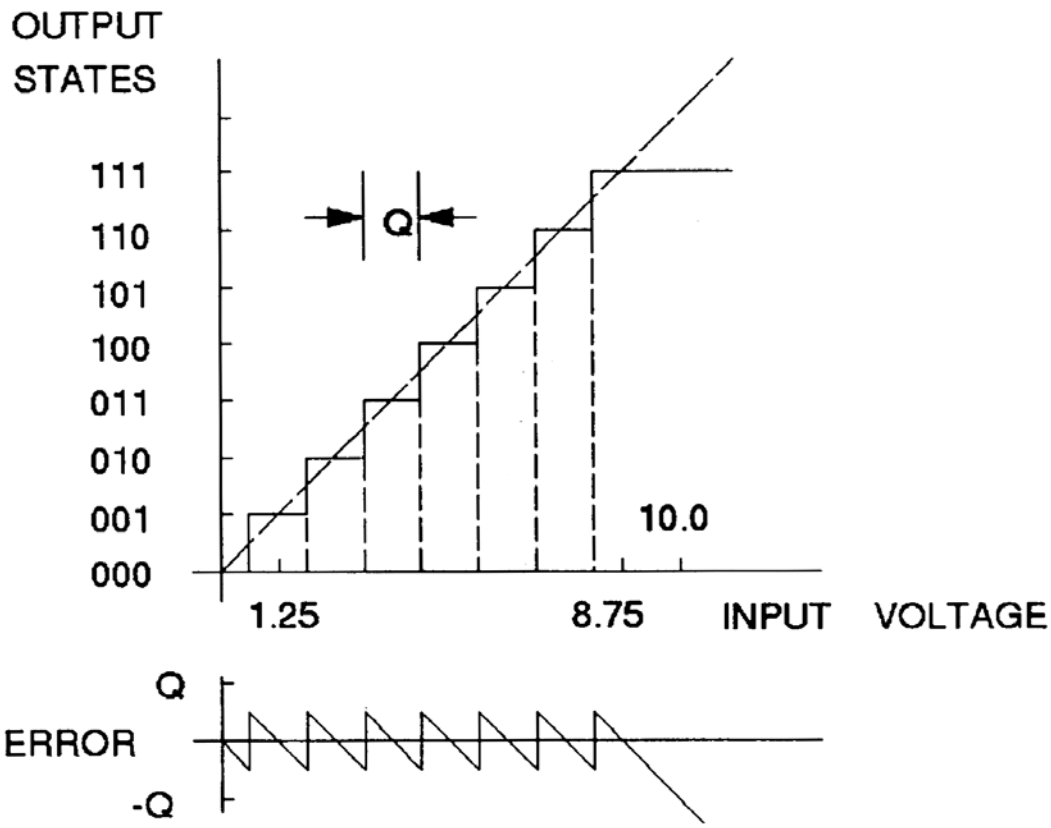
Other principles are:

1. Usage of reference voltages
2. Weighted inputs

ACCURACY AND DYNAMIC RANGE

$$Q = LSB\ value = \frac{full\ scale\ voltage}{2^n}$$

LSB value... voltage at least significant bit (e.g. 2.5 μV)
n... bits of resolution



s0604

If full scale voltage of 2.5 V is used in connection with Q=2.5 μV, we get n = 20 bits of resolution (which is more than 16 bit and less than 32 bit).

Dynamic range:

$$D = 20\log\left(\frac{A_{\max}}{A_{\min}}\right)\ ;\longrightarrow[dB]$$

$$D = 20\log(2^n - 1)$$

$$\mathrm{n} = 16 \longrightarrow \mathrm{D} = 96\ \mathrm{dB}$$

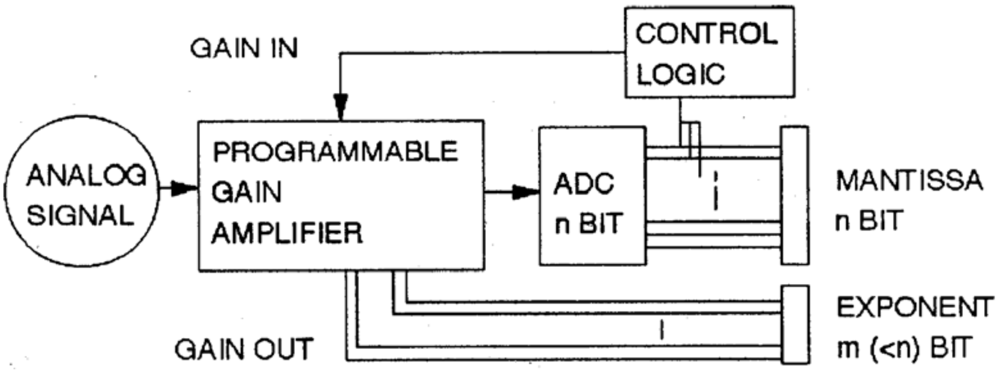
$$\mathrm{n} = 32 \longrightarrow \mathrm{D} = 193\ \mathrm{dB}$$

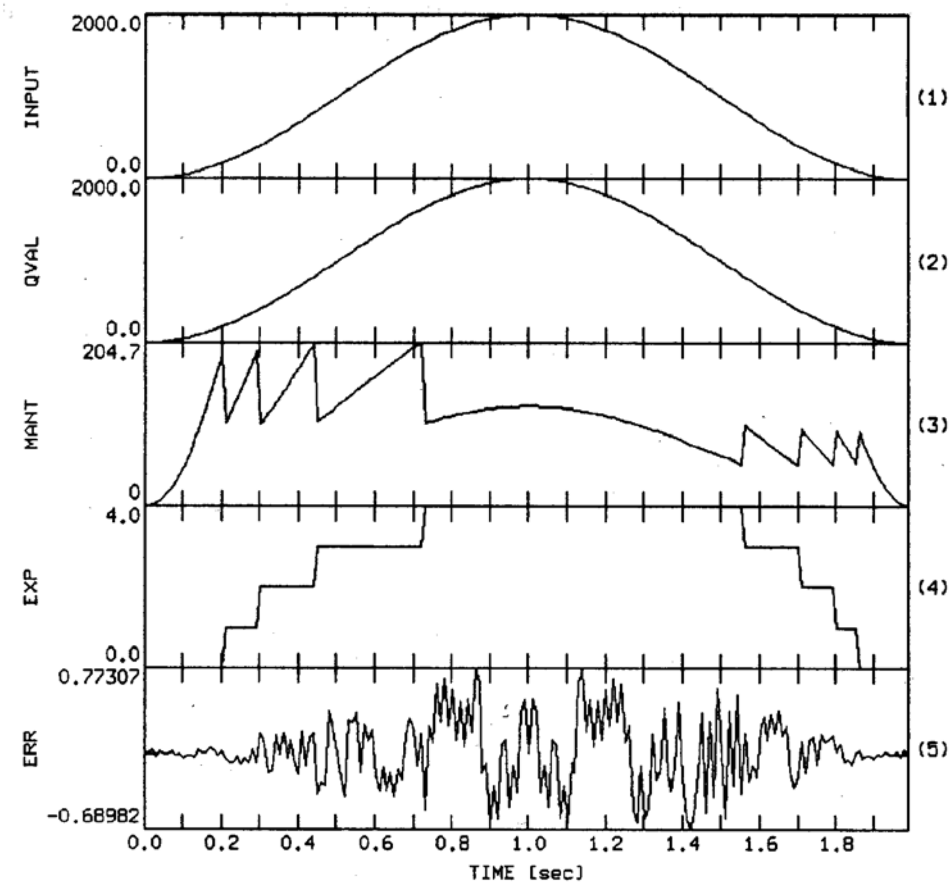
GAIN RANGING

$$D = 20\log((2^n - 1)\ 2^{(2^m - 1)}) \longrightarrow [dB]$$

$$\approx 20\log(2^{n+2^m})$$

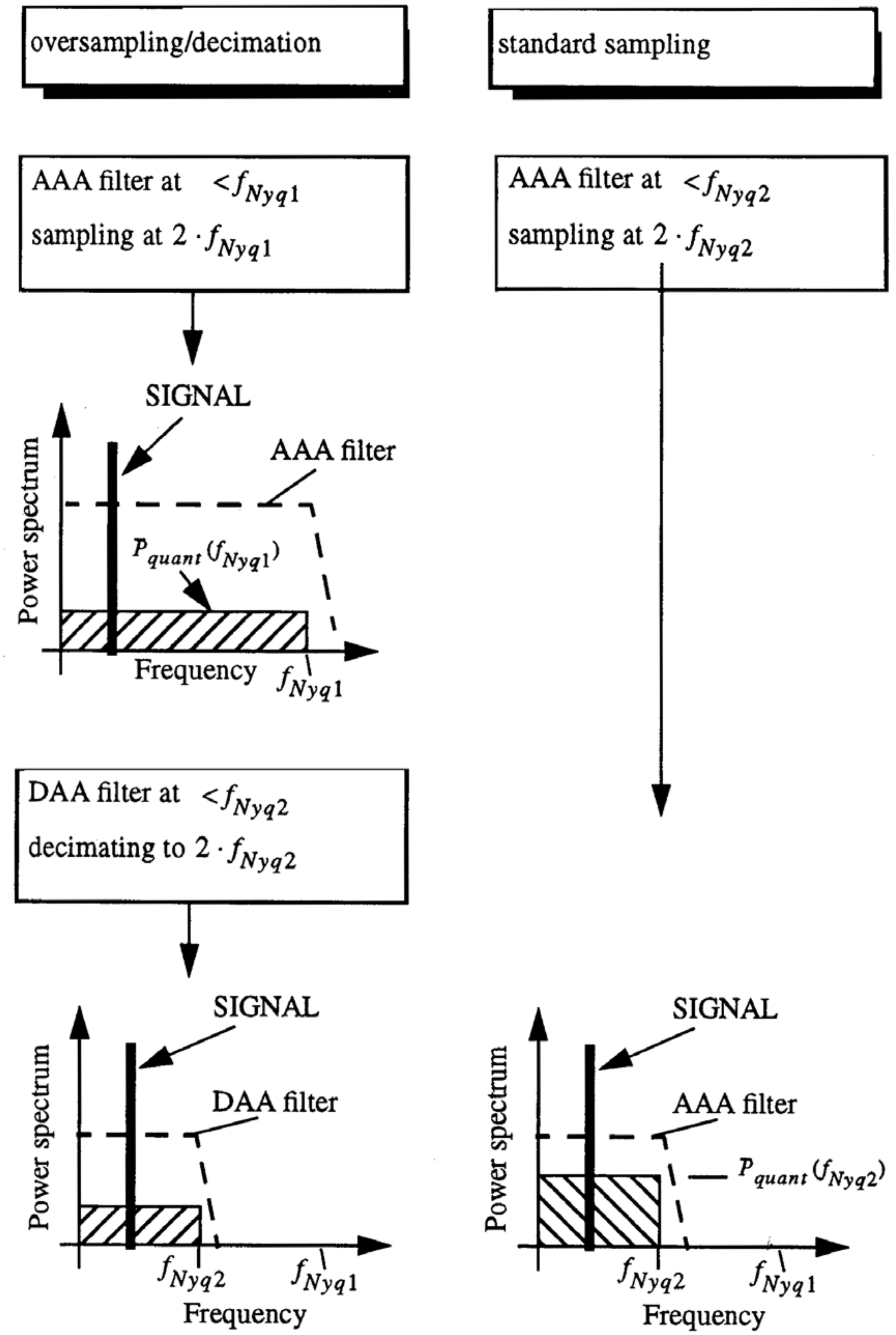
$$\mathrm{n} = 12, \mathrm{m} = 4 \longrightarrow \mathrm{D} = 168\ \mathrm{dB}$$





s0608 and s0610

OVERSAMPLING AND DECIMATION



s0612

(see Scherbaum, F. 1996)